

Ω versus graviphoton

Ahmad Zein Assi

*Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5,
CH-3012 Bern, Switzerland*

Received 5 December 2018; received in revised form 7 April 2019; accepted 12 May 2019

Available online 15 May 2019

Editor: Stephan Stieberger

Abstract

In this work, I study the deformation of the topological string by $\bar{\Omega}$, the complex conjugate of the Ω -deformation. Namely, I identify $\bar{\Omega}$ in terms of a physical state in the string spectrum and verify that the deformed Yang-Mills and ADHM actions are reproduced. This completes the study initiated in [1] where we show that $\bar{\Omega}$ decouples from the one-loop topological amplitudes in heterotic string theory. Similarly to the $\mathcal{N} = 2^*$ deformation, I show that the quadratic terms in the effective action play a crucial role in obtaining the correct realisation of the full Ω -deformation. Finally, I comment on the differences between the graviphoton and the Ω -deformation in general and discuss possible $\bar{\Omega}$ remnants at the boundary of the string moduli space.

© 2019 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

1. Introduction

Little attention has been devoted to the study of the Ω -background as a non-holomorphic deformation at the string level. Indeed, from the gauge theory point of view, the Ω -deformation is a background that twists space-time in a particular way allowing for a path-integral derivation of the partition function through localisation [2–4]. Viewed as a background, it is non-holomorphic in the sense that it has a holomorphic as well as an anti-holomorphic part denoted $\bar{\Omega}$. However, the latter decouples from physical observables and this is clear from the localisation perspective where $\bar{\Omega}$ is a Q-exact deformation of the effective action.

E-mail address: zeinassi@cern.ch.

From the string theory point of view, there is a natural *topological* limit that one can impose on the Ω -background. If one denotes $\epsilon_{1,2}$ the two parameters of the latter, then, for $\epsilon_1 + \epsilon_2 = 0$ the Ω -deformed gauge theory partition function is the field theory limit of the topological string partition function F_g [5] which computes a class of higher derivative gravitational couplings in the effective action [6]. The connection between string amplitudes and supersymmetric gauge theories has been further extended beyond topological limit [7–9]. In all these studies, the anti-holomorphic part $\bar{\Omega}$ is implicitly set to zero. This is understandable from the gauge theory side because of the decoupling of $\bar{\Omega}$. In string theory, however, it is not clear, *a priori*, that the same property remains true. A particular instance where the breakdown, in string theory, of gauge theory properties is the moduli dependence of topological amplitudes. Indeed, as shown in [10], the topological amplitudes F_g have a non-holomorphic dependence on the moduli of the string compactification stemming from the boundary of the moduli space only while, in gauge theory, F_g is purely holomorphic.

The purpose of the present paper is to initiate a similar analysis for the full Ω -deformation (including $\bar{\Omega}$) in the topological limit. The goal is to study the possible dependence in string theory on $\bar{\Omega}$ by identifying it in the string theory spectrum. In [1], by making a particular ansatz for $\bar{\Omega}$, we showed that $\bar{\Omega}$ decouples from the topological amplitudes F_g perturbatively. Here, I prove that this ansatz is correct by showing that it leads to the correct effective actions in field theory. This is done by coupling the open string degrees of freedom of a Dp-D(p+4) system in type I string theory to the closed string background using similar techniques as in [8,11,12].

The paper is organised as follows. In Section 2, I review the ansatz of [1] in the context of type I string theory compactified on a $T^2 \times K3$ orientifold and then, in Section 3, I derive the perturbative (Yang-Mills) and non-perturbative (ADHM) effective actions in the presence of the string theory $\Omega, \bar{\Omega}$ -background. Finally, in Section 4, I briefly review the results of [1] for completeness. To keep the discussion clear, several technical aspects and useful results are presented in two appendices.

2. The stringy $\bar{\Omega}$

In [1], starting from heterotic string theory compactified on $T^2 \times K3$, we postulated that $\bar{\Omega}$ can be identified as a constant background for a self-dual field strength F_T which is the vector partner of the Kähler modulus T of T^2 . Here, I repeat the same analysis in the context of the dual type I string theory.

Recall that, in ten dimensions, type I and heterotic string theories are S-dual to each other. However, in four dimensions, there are regions in the moduli space where both theories are weakly coupled [13]. Hence, I focus on those particular regions and consider a type I theory compactified on $T^2 \times K3$. Here, I realise $K3$ as a T^4/\mathbb{Z}_2 orientifold which admits both D9- and D5-branes. Tadpole cancellation restricts the number of such D-branes. However, I keep their number generic as one could consider a non-compact $K3$ as well. In Table 1, I summarise the mapping between the universal scalar fields of the vector multiplet moduli space.

Notice that the axion-dilaton is mapped to one another [13,14]. Furthermore, the complex structure modulus U of T^2 is unchanged and the Kähler modulus T in Heterotic is mapped to another dilaton-like field denoted S' . Indeed, the presence of two dilatons in Type I is not surprising since the theory contains D9- and D5-branes whose coupling constants are given by the imaginary parts of two different fields, namely S and S' respectively.

In order to identify the $\bar{\Omega}$ deformation in terms of the type I physical string spectrum, I focus on the universal vector multiplet sector, *i.e.* the so-called STU-model. Namely, the three vector

Table 1
Mapping of the universal fields
under Heterotic/Type I duality
in four dimensions.

Heterotic	Type I
S	S
T	S'
U	U

multiplet moduli are S, S' and U. To keep the discussion clear, I do not consider Wilson lines. In addition to the three vector fields associated to each scalar, there is another vector field stemming from the $\mathcal{N} = 2$ gravity multiplet, the graviphoton, so that there are four gauge fields denoted F_I with $I = G, S, S', U$.

Since the question under consideration is the realisation of the Ω -deformation, recall that one can construct it geometrically as a non-trivial T^2 fibration over space-time, in such a way that, when one goes around the cycles of T^2 , the space-time fields are rotated with an arbitrary angle. This geometric picture of a reduction from six dimensions on the metric with line element

$$ds^2 = g_{\mu\nu} (dX^\mu + \Omega^\mu{}_\rho X^\rho dZ + \bar{\Omega}^\mu{}_\rho X^\rho d\bar{Z}) (dX^\nu + \Omega^\nu{}_\rho X^\rho dZ + \bar{\Omega}^\nu{}_\rho X^\rho d\bar{Z}) + ds_{T^2}^2 \quad (2.1)$$

is very helpful. Here, $\mu, \nu \in \llbracket 0, 3 \rrbracket$ are space-time indices, Z is the complexified T^2 coordinate and $\Omega, \bar{\Omega}$ are the space-time rotation matrices depending on two complex parameters $\epsilon_{1,2}$. Hence, the Lorentz group $SU(2)_L \times SU(2)_R$ is explicitly broken. For practical purposes, redefine them as

$$\epsilon_\pm = \frac{\epsilon_1 \pm \epsilon_2}{2}. \quad (2.2)$$

For the present matter, I work in the topological limit $\epsilon_+ = 0$. From the seminal work [6], one can show that the holomorphic part of the Ω -deformation can be described as a constant background for the self-dual part of the graviphoton field strength. The self-duality condition means that $\epsilon_- \equiv \epsilon$ is sensitive to $SU(2)_L$ only. In this convention, for $\epsilon_+ = 0$, $SU(2)_R$ remains unbroken. Consequently, in order to describe $\bar{\Omega}$ in type I string theory, one should turn on a constant background for the self-dual field strength of one of the following fields.

1. $F_{S'}^{(-)}$: $V_T = \epsilon_\mu \partial X^\mu \bar{\partial} \bar{Z} + \text{RR}$.
2. $F_U^{(-)}$: $V_U = \epsilon_\mu \partial X^\mu \bar{\partial} Z + \text{RR}$.
3. $F_S^{(-)}$: $V_S = \epsilon_\mu \partial \bar{Z} \bar{\partial} X^\mu + \text{RR}$.

The superscript $(-)$ stresses the fact that the field strengths are self-dual. Notice that I have only written the bosonic part of the vertex operators which should be completed by the fermionic and the Ramond-Ramond parts.¹ As opposed to heterotic, the latter arises typically in orientifold compactifications since the supercharges are combinations of left- and right-moving ones. In addition, I have implicitly assumed that the polarisation ϵ_μ is chosen in such a way to satisfy the self-duality constraint. Recall that the graviphoton vertex operator is

$$V_G = \epsilon_\mu \partial Z \bar{\partial} X^\mu + \text{RR}. \quad (2.3)$$

¹ The full vertex operators are written in Section 3.

Consequently, one is led to choosing $F_{S'}^{(-)}$ and $F_S^{(-)}$ as natural candidates for $\bar{\Omega}$. Here, ‘natural’ means that $\bar{\Omega}$ is understood, in some sense, as the complex conjugate of Ω .

In order to rigorously make the selection, I first consider the effect of each of these states on the gauge theory degrees of freedom. More precisely, following the ideas pioneered in [15] and further exploited in [8,11,12], I analyse the possible couplings of the field-strengths with the gauge theory degrees of freedom (including instantons) realised as the massless excitations of the open strings in Dp-D(p+4) branes systems. This is naturally realised in the dual type I string theory [8] studied in the following section.

3. $\bar{\Omega}$ -deformed effective actions

3.1. Vertex operators

In the present section, I use the D9-branes to realise the gauge theory. By T-duality, this is equivalent to the D3-branes realisation [15], see also [16] for a review and complete analysis. In this setup, gauge theory instantons are realised in terms of D5-branes called D5-instantons wrapping $T^2 \times K3$. This configuration is summarised in the following table.

brane	num.	X^0	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9
D9	N	•	•	•	•	•	•	•	•	•	•
D5	k					•	•	•	•	•	•

$\underbrace{\hspace{10em}}_{\text{space-time} \sim \mathbb{R}^4}$
 $\underbrace{\hspace{4em}}_{T^2}$
 $\underbrace{\hspace{10em}}_{K3 \sim T^4/\mathbb{Z}_2}$

In this picture, the massless excitations of the open strings stretched between the D9-branes are the gauge theory degrees of freedom whereas the massless excitations of the open strings with at least one endpoint on a D5-instanton correspond to the ADHM moduli. Hence, besides the D9-branes excitations that I refer to as the 9-9 sector, there are two classes of open string excitations in the instanton sector, depending on whether the location of one endpoint of the open string only lies on a D5-instanton (9-5, 5-9 or mixed sector) or both (5-5 or unmixed sector). The ADHM moduli are non dynamical fields due to the Dirichlet boundary conditions in all transverse directions. These fields are summarised in Table 2.

In the 9-9 sector, the massless excitations consist of a number of $\mathcal{N} = 2$ vector multiplets, each of which containing a vector field A^μ , a complex scalar ϕ as well as four gaugini ($\Lambda^{\alpha A}, \Lambda_{\dot{\alpha} A}$) that are in the $(\mathbf{2}, \mathbf{1})$ representation of $SU(2)_+ \times SU(2)_-$. The bosonic degrees of freedom stem from the NS sector, while the fermionic ones from the R sector. These fields, taken separately, realise a Yang-Mills theory in four dimensions. Denoting ψ^μ, Ψ the fermionic partners of X^μ, Z respectively and φ the free boson that bosonises the superghost, their vertex operators are

$$V_A(z) = \frac{A_\mu(p)}{\sqrt{2}} \psi^\mu(z) e^{ip \cdot X(z)} e^{-\varphi(z)}, \quad V_\phi(z) = \frac{\phi(p)}{\sqrt{2}} \Psi(z) e^{ip \cdot X(z)} e^{-\varphi(z)}, \quad (3.1)$$

$$V_\Lambda(z) = \Lambda^{\alpha A} S_\alpha(z) S_A(z) e^{ip \cdot X(z)} e^{-\frac{1}{2}\varphi(z)}, \quad V_{\bar{\Lambda}}(z) = \Lambda_{\dot{\alpha} A} S^{\dot{\alpha}}(z) S^A(z) e^{ip \cdot X(z)} e^{-\frac{1}{2}\varphi(z)}. \quad (3.2)$$

The spin fields S are defined in Appendix A. In the 5-5 and NS sector, there are six bosonic moduli that can be written as a real vector a^μ and a complex scalar χ . From the perspective of the gauge theory living on the world-volume of the D9-branes, a^μ parametrises the position of

Table 2

Summary of the massless spectrum of the D5/D9 system and the decomposition of the scalar fields. I display the transformation properties under the groups $SO(4)_{ST} \times SU(2)_+ \times SU(2)_-$, while c_{T^2} is the charge under $SO(2)_{T^2}$.

sector	field	$SO(4)_{ST}$	c_{T^2}	$SU(2)_+ \times SU(2)_-$
9-9	A^μ	$(1/2, 1/2)$	0	$(1, 1)$
	$\Lambda^{\alpha A}$	$(1/2, 0)$	1/2	$(2, 1)$
	$\Lambda_{\dot{\alpha} A}$	$(0, 1/2)$	-1/2	$(2, 1)$
	ϕ	$(0, 0)$	-1	$(1, 1)$
5-5	a^μ	$(1/2, 1/2)$	0	$(1, 1)$
	χ	$(0, 0)$	-1	$(1, 1)$
	$M^{\alpha A}$	$(1/2, 0)$	1/2	$(2, 1)$
	$\lambda_{\dot{\alpha} A}$	$(0, 1/2)$	-1/2	$(2, 1)$
5-9	$\omega_{\dot{\alpha}}$	$(0, 1/2)$	0	$(1, 1)$
	μ^A	$(0, 0)$	1/2	$(2, 1)$

gauge theory instantons. In the Ramond sector, there are eight fermionic moduli $M^{\alpha A}$, $\lambda_{\dot{\alpha} A}$. The vertex operators of these states are

$$V_a(z) = g_6 a_\mu \psi^\mu(z) e^{-\varphi(z)}, \quad V_\chi(z) = \frac{\chi}{\sqrt{2}} \Psi(z) e^{-\varphi(z)}, \quad (3.3)$$

$$V_M(z) = \frac{g_6}{\sqrt{2}} M^{\alpha A} S_\alpha(z) S_A(z) e^{-\frac{1}{2}\varphi(z)}, \quad V_\lambda(z) = \lambda_{\dot{\alpha} A} S^{\dot{\alpha}}(z) S^A(z) e^{-\frac{1}{2}\varphi(z)}. \quad (3.4)$$

Here, g_5 is the D5-brane coupling constant. For a Dp-brane, it is given by

$$g_{p+1}^2 = 4\pi(2\pi\sqrt{\alpha'})^{p-3} g_s. \quad (3.5)$$

Finally, let's consider the mixed moduli. From the NS sector, the fermionic coordinates give rise to two Weyl spinors $(\omega_{\dot{\alpha}}, \bar{\omega}_{\dot{\alpha}})$ of $SO(4)$. They have the same chirality due to the specific choice of boundary conditions of the D5-branes. In addition, they describe the size of the instanton. In the R sector, one gets two Weyl fermions $(\mu^A, \bar{\mu}^A)$ transforming in the fundamental representation of $SO(6)$. The vertex operators for the mixed sectors contain the *twist operators* that change the coordinates boundary conditions from Dirichlet to Neumann and vice versa. These are bosonic fields $\Delta, \bar{\Delta}$ of conformal dimension 1/4. The vertex operators are

$$V_\omega(z) = \frac{g_1}{\sqrt{2}} \omega_{\dot{\alpha}} \Delta(z) S^{\dot{\alpha}}(z) e^{-\varphi(z)}, \quad V_{\bar{\omega}}(z) = \frac{g_1}{\sqrt{2}} \bar{\omega}_{\dot{\alpha}} \bar{\Delta}(z) S^{\dot{\alpha}}(z) e^{-\varphi(z)} \quad (3.6)$$

$$V_\mu(z) = \frac{g_1}{\sqrt{2}} \mu^A \Delta(z) S_A(z) e^{-\frac{1}{2}\varphi(z)}, \quad V_{\bar{\mu}}(z) = \frac{g_1}{\sqrt{2}} \bar{\mu}^A \bar{\Delta}(z) S_A(z) e^{-\frac{1}{2}\varphi(z)}. \quad (3.7)$$

This summarises the complete description of the gauge theory states in terms of CFT vertex operators. In order to study the $\bar{\Omega}$ -deformation, one must include the closed string operators describing it. As mentioned above, the preserved supercharges are a combination of the left- and right-moving supercharges due to the orientifold action. This implies that the vertex operators for the vector fields have, generically, an NS-NS and a R-R part each. They can be derived explicitly by spectral flow from the universal scalars' vertex operators. In [7], the self-dual graviphoton and U-vector field operators were shown to take the form

$$V^\pm(y, \bar{y}) = \frac{1}{8\pi\sqrt{2}} F_{\mu\nu}^\pm \left[\psi^\mu \psi^\nu(y) e^{-\varphi(\bar{y})} \Psi(\bar{y}) + e^{-\varphi(y)} \Psi(y) \psi^\mu \psi^\nu(\bar{y}) \right. \\ \left. \pm \frac{i}{2} e^{-\frac{1}{2}(\varphi(y)+\varphi(\bar{y}))} S_\alpha(y) (\sigma^{\mu\nu})^{\alpha\beta} S_\beta(\bar{y}) \epsilon^{AB} S_A(y) S_B(\bar{y}) \right], \quad (3.8)$$

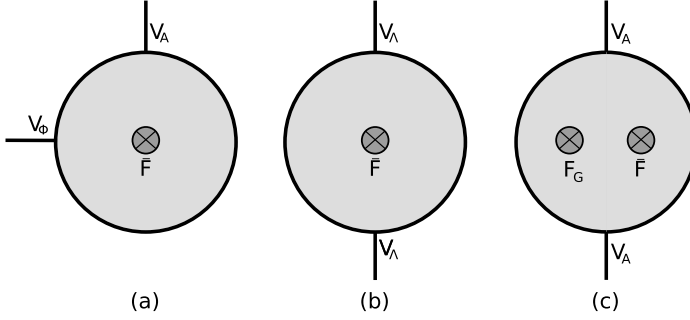


Fig. 1. Disc diagrams in the 9-9 sector. Diagram (a) involves two bosonic boundary insertions and one bulk insertion of \bar{F} , diagram (b) two fermionic insertions and one \bar{F} , whereas diagram (c) has two bosonic boundary insertions and two bulk insertions of a graviphoton and an \bar{F} .

with $V_G = V^-$, $V_U = V^+$, $F^- = F_G^{(-)}$ and $F^+ = F_U^{(-)}$. Also note that Ψ is the worldsheet fermion in the T^2 direction. It is important to notice that the difference between the two states is merely the sign between the NS-NS and R-R parts. Similarly, the vertex operators for the candidate states, *i.e.* $F_S^{(-)}$ and $F_{S'}^{(-)}$ are found to be

$$\begin{aligned} \bar{V}^\pm(y, \bar{y}) = \frac{1}{8\pi\sqrt{2}} \bar{F}_{\mu\nu}^\pm \left[\psi^\mu \psi^\nu(y) e^{-\varphi(\bar{y})} \bar{\Psi}(\bar{y}) + e^{-\varphi(y)} \bar{\Psi}(y) \psi^\mu \psi^\nu(\bar{y}) \right. \\ \left. \pm \frac{i}{2} e^{-\frac{1}{2}(\varphi(y)+\varphi(\bar{y}))} S_\alpha(y) (\sigma^{\mu\nu})^{\alpha\beta} S_\beta(\bar{y}) \epsilon^{\hat{A}\hat{B}} S_{\hat{A}}(y) S_{\hat{B}}(\bar{y}) \right], \end{aligned} \quad (3.9)$$

where $V_S = \bar{V}^+$, $V_{S'} = \bar{V}^-$, $\bar{F}^+ = F_S^{(-)}$ and $\bar{F}^- = F_{S'}^{(-)}$.

3.2. Deformed Yang-Mills action

I now calculate the deformations to the Yang-Mills effective action due to the candidates for the $\bar{\Omega}$ -deformation. In order to achieve this, I calculate all possible tree-level couplings between the self-dual S- and S'-vectors to leading order in α' . In the type I theory under study, these are simply disc diagrams whose boundary lies entirely on a D9-brane. In addition, I insert at least one self-dual S- or S'-vector vertex operator in the bulk of the disc diagram (and possibly some graviphotons) while including an arbitrary number of open string insertions at the boundary from the 9-9 sector. In fact, the number of such insertions is highly restricted by the fact that I am only interested in the gauge theory action.

From a practical point of view, to avoid repeating the calculations twice, I split the vertex operator for the $\bar{\Omega}$ -deformation into an NS-NS part V_1 and a R-R one V_2 . It turns out that only few disc amplitudes are potentially non-vanishing in the field theory limit and some of those are summarised in Fig. 1.

I start by evaluating the amplitude with the gaugini. Notice that, due to the specific self-duality of the closed string vertex, the only possible non-trivial couplings are with $\Lambda^{\alpha A}$. Furthermore, had one used the graviphoton vertex instead, the amplitude would have vanished trivially by the non-conservation of the $SO(2)_{T^2}$ -charge. Using the doubling trick, I convert the disc into the full plane with a \mathbb{Z}_2 involution. I then split the closed string vertices into their left- and right-moving parts which can be considered as independent. As such, the tree-point disc amplitude becomes effectively a four-point amplitude. In addition, I soak up c -ghost zero-modes on the sphere by

attaching c to three dimension-one operators in such a way that the resulting vertex operator is BRST closed. In order to avoid subtleties due to the use of unintegrated vertex operators in different ghost pictures as discussed, for instance, in [16], I don't attach the c -ghost to operators in the zero-picture. The contribution from the NS-NS part of \bar{F} is

$$\begin{aligned} \left\langle \left\langle c V_{\Lambda}(z_1) c V_{\Lambda}(z_2) c V_{\bar{F}}^{(-1,0)}(z_3, z_4) \right\rangle \right\rangle &= \frac{1}{8\pi\sqrt{2}} \Lambda^{\alpha A} \Lambda^{\beta B} \bar{F}_{\mu\nu} \left\langle \left\langle c(z_1) S_{\alpha}(z_1) S_A(z_1) e^{-\frac{1}{2}\varphi(z_1)} \right. \right. \\ &\quad \times c(z_2) S_{\beta}(z_2) S_B(z_2) e^{-\frac{1}{2}\varphi(z_2)} \\ &\quad \left. \left. \times c(z_3) e^{-\varphi(z_3)} \bar{\Psi}(z_3) \psi^{\mu} \psi^{\nu}(z_4) \right\rangle \right\rangle + (\text{Left} \leftrightarrow \text{Right}) . \end{aligned} \quad (3.10)$$

Notice that I have denoted the left and right parts of the NS-NS vertex by two different labels. The double brackets notation means that one should integrate over $n-3$ free positions with n being the total number of inserted points. In (3.10), one must integrate over z_4 only. Performing the contractions between the various operators using the standard results summarised in Appendix A, I find

$$\left\langle \left\langle c V_{\Lambda}(z_1) c V_{\Lambda}(z_2) c V_{\bar{F}}^{(-1,0)}(z_3, z_4) \right\rangle \right\rangle = -\frac{i}{16\pi\sqrt{2}} \epsilon_{AB} (\sigma^{\mu\nu})_{\alpha\beta} \Lambda^{\alpha A} \Lambda^{\beta B} \bar{F}_{\mu\nu} \int dz_4 \frac{z_{12}}{z_{14} z_{24}} . \quad (3.11)$$

I have used the notation $z_{ij} \equiv z_i - z_j$. The integral over z_4 is straightforward. I identify $\bar{\epsilon}$ using a particular normalisation for $\bar{F}_{\mu\nu}$:

$$\bar{F}_{\mu\nu} \equiv \eta_{\mu\nu}^3 \frac{\bar{\epsilon}}{2} , \quad (3.12)$$

with $\eta_{\mu\nu}^c$ being the 't Hooft symbols. Thus, I obtain

$$\left\langle \left\langle c V_{\Lambda}(z_1) c V_{\Lambda}(z_2) c V_{\bar{F}}^{(-1,0)}(z_3, z_4) \right\rangle \right\rangle = -i \frac{\bar{\epsilon}}{\sqrt{2}} \Lambda^{\alpha A} \Lambda^{\beta B} A(\sigma^3)_{\alpha\beta} . \quad (3.13)$$

I now turn to the contribution of the R-R piece of the closed string operator which is

$$\begin{aligned} &\left\langle \left\langle c V_{\Lambda}(z_1) c V_{\Lambda}(z_2) c V_{\bar{F}}^{(-\frac{1}{2}, -\frac{1}{2})}(z_3, z_4) \right\rangle \right\rangle \\ &= \frac{i}{16\pi\sqrt{2}} \Lambda^{\alpha A} \Lambda^{\beta B} \bar{F}_{\mu\nu} (\sigma^{\mu\nu})^{\gamma\delta} \epsilon^{\hat{C}\hat{D}} \left\langle \left\langle c(z_1) S_{\alpha}(z_1) S_A(z_1) e^{-\frac{1}{2}\varphi(z_1)} \right. \right. \\ &\quad \times c(z_2) S_{\beta}(z_2) S_B(z_2) e^{-\frac{1}{2}\varphi(z_2)} c(z_3) e^{-\frac{1}{2}\varphi(z_3)} S_{\lambda}(z_3) S_{\hat{C}}(z_3) \\ &\quad \left. \left. \times e^{-\frac{1}{2}\varphi(z_3)} S_{\delta}(z_4) S_{\hat{D}}(z_4) \right\rangle \right\rangle + (\text{Left} \leftrightarrow \text{Right}) . \end{aligned} \quad (3.14)$$

After making all possible contractions, the result is

$$\begin{aligned} &\left\langle \left\langle c V_{\Lambda}(z_1) c V_{\Lambda}(z_2) c V_{\bar{F}}^{(-\frac{1}{2}, -\frac{1}{2})}(z_3, z_4) \right\rangle \right\rangle \\ &= -\frac{i}{16\pi\sqrt{2}} \Lambda^{\alpha A} \Lambda^{\beta B} \bar{F}_{\mu\nu} (\sigma^{\mu\nu})^{\gamma\delta} \epsilon^{\hat{C}\hat{D}} \epsilon_{\alpha\delta} \epsilon_{\beta\gamma} \epsilon_{AB} \epsilon_{\hat{C}\hat{D}} \int dz_4 \frac{z_{12}}{z_{14} z_{24}} . \end{aligned} \quad (3.15)$$

Therefore, one gets exactly the same result as for the NS-NS contribution:

$$\left\langle\left\langle c V_{\Lambda}(z_1) c V_{\Lambda}(z_2) c V_{\bar{F}}^{(-\frac{1}{2}, -\frac{1}{2})}(z_3, z_4) \right\rangle\right\rangle = -i \frac{\bar{\epsilon}}{\sqrt{2}} \Lambda^{\alpha A} \Lambda^{\beta}{}_{\alpha} (\sigma^3)_{\alpha\beta}. \quad (3.16)$$

This is to be expected based on supersymmetry arguments. One now puts both results together and notices a clear difference between the S - and the S' -vectors. Namely, for the F_S one has

$$\langle V_{\Lambda} V_{\Lambda} V_S \rangle = 0 \quad (3.17)$$

while for $F_{S'}$

$$\langle V_{\Lambda} V_{\Lambda} V_{S'} \rangle = -i \sqrt{2} \bar{\epsilon} \Lambda^{\alpha A} \Lambda^{\beta}{}_{\alpha} (\sigma^3)_{\alpha\beta}. \quad (3.18)$$

In fact, to the leading order I am concerned by, there are no further couplings between the closed string vertices and the fermionic fields.

This is already a clear indication that $\bar{\Omega}$ is realised by the S' -vector as I now confirm from the following calculations. Note that this type of cancellations was already observed in a similar setup in [8] where the problem of the refinement is addressed.

Now consider the possible couplings with the bosonic open string degrees of freedom. The first non-trivial coupling corresponds to diagram (a) in Fig. 1. As before, I split the closed string vertex into its NS-NS and R-R parts. The setup for the vertex operators is as follows.

$$\begin{aligned} V_{\phi}(z_1) &= \frac{\phi}{\sqrt{2}} c e^{-\varphi} \Psi e^{2ip_1 \cdot X}(z_1), \\ V_A(z_2) &= A_{\mu} (\partial X^{\mu} - 2i(p_2 \cdot \psi) \psi^{\mu}) e^{2ip_2 \cdot X}(z_2), \\ V_{\bar{F}}(z_3) &= c e^{-\frac{\varphi}{2}} S_{\alpha} \hat{A} e^{i\bar{P} \cdot X}(z_3), \\ V_{\bar{F}}(z_4) &= \frac{i}{16\pi\sqrt{2}} \epsilon^{\hat{A}\hat{B}} (\sigma^{\lambda\nu})^{\alpha\beta} \bar{F}_{\lambda\nu} e^{-\frac{\varphi}{2}} S_{\beta} \hat{B} e^{i\bar{P} \cdot X}(z_4). \end{aligned} \quad (3.19)$$

Hence, I choose to integrate over z_2 only. Notice that only the fermion bilinear term of the gauge field can lead to a non-vanishing amplitude. In addition, one can already set the momenta of the exponentials to zero since this amplitude turns out not to stem from contact terms, *i.e.* the integrals over the positions are convergent for vanishing momenta. The correlator is thus

$$\begin{aligned} \left\langle V_{\phi}(z_1) V_A(z_2) V^{(-\frac{1}{2}, -\frac{1}{2})}(z_3, z_4) \right\rangle &= \frac{1}{16\pi} \epsilon^{\hat{A}\hat{B}} \phi p_{2\rho} A_{\mu} \bar{F}^{\alpha\beta} \frac{z_{13} z_{14} z_{24}}{z_{13}^{1/2} z_{14}^{1/2} z_{34}^{1/4}} \\ &\quad \times \langle \psi(z_1) \psi^{\rho} \psi^{\mu}(z_2) S_{\alpha} \hat{A}(z_3) S_{\beta} \hat{B}(z_4) \rangle, \end{aligned} \quad (3.20)$$

where I have introduced the notation $F^{\alpha\beta} \equiv (\sigma^{\mu\nu})^{\alpha\beta} F_{\mu\nu}$. Using the results of Appendix A, one can calculate the CFT correlator and find

$$\left\langle V_{\phi}(z_1) V_A(z_2) V^{(-\frac{1}{2}, -\frac{1}{2})}(z_3, z_4) \right\rangle = -\frac{i}{16\pi} \phi p_{2\rho} A_{\mu} \bar{F}^{\alpha\beta} (\sigma^{\rho\mu})_{\alpha\beta} \frac{z_{34}}{z_{23} z_{24}}. \quad (3.21)$$

Integrating over the disc, one obtains

$$\left\langle\left\langle V_{\phi}(z_1) V_A(z_2) V^{(-\frac{1}{2}, -\frac{1}{2})}(z_3, z_4) \right\rangle\right\rangle = \phi F_{\mu\nu} \bar{F}^{\mu\nu}. \quad (3.22)$$

Including the NS-NS part, the total contribution to the action becomes

$$\langle V_{\phi} V_A V_{S'} \rangle = \bar{\epsilon} \phi F_{\mu\nu} \eta^{3\mu\nu}. \quad (3.23)$$

Finally, I analyse the possible quadratic terms in the Ω -deformation. I first consider possible mass terms for the gauge fields, namely I calculate the coupling between two gauge fields, one graviphoton and one S' -vector,

$$\langle V_A(z_1)V_A(z_2)V_G(z_3,z_4)V_{S'}(z_5,z_6) \rangle, \quad (3.24)$$

and choose for convenience to fix the positions z_1, z_2 and z_4 to $-\infty, 1$ and $-ix$ with x being real (so that $z_3 = ix$). This is done by attaching a c -ghost to the corresponding vertex operators. In order to keep the calculation simple, I take the gauge fields' vertex operators in the 0-ghost picture. However, by BRST invariance, this requires including an additional contribution to the vertex (3.1) as discussed in [8,16]. Hence, the vertex operators are explicitly given by

$$V_A(z_1) = A_\mu \left[c(\partial X^\mu + 2i p_{1,\nu} \psi^\mu \psi^\nu) - \gamma \psi^\mu \right] e^{2ip_1 \cdot X}(z_1), \quad (3.25)$$

$$V_A(z_2) = A_\rho \left[c(\partial X^\rho + 2i p_{2,\sigma} \psi^\rho \psi^\sigma) - \gamma \psi^\rho \right] e^{2ip_2 \cdot X}(z_2), \quad (3.26)$$

$$V_G(z_3) = c e^{-\frac{\varphi}{2}} S_\alpha S_A e^{iP \cdot X}(z_3), \quad (3.27)$$

$$V_G(z_4) = -\frac{i}{16\pi\sqrt{2}} \epsilon^{AB} (\sigma^{\lambda\kappa})^{\alpha\beta} F_{\lambda\kappa} e^{-\frac{\varphi}{2}} S_\beta S_B e^{iP \cdot X}(z_4), \quad (3.28)$$

$$V_{\bar{F}}(z_5) = c e^{-\frac{\varphi}{2}} S_\gamma S_{\hat{A}} e^{i\bar{P} \cdot X}(z_5), \quad (3.29)$$

$$V_{\bar{F}}(z_6) = \frac{i}{16\pi\sqrt{2}} \epsilon^{\hat{A}\hat{B}} (\sigma^{\omega\xi})^{\gamma\delta} \bar{F}_{\omega\xi} e^{-\frac{\varphi}{2}} S_\delta S_{\hat{B}} e^{i\bar{P} \cdot X}(z_6). \quad (3.30)$$

Here, I have only written the R-R parts of the closed string vertices whose contributions are considered first. Furthermore, recall that momentum conservation along the Neumann directions implies that $p_1 + p_2 + P + \bar{P} = 0$. Even though I compute a coupling at a fixed momentum order, I keep the momenta generic in order to consistently regularise the worldsheet integrals. Equivalently, the contribution of interest arises as a contact term of the form $(p_i \cdot p_j)/(p_i \cdot p_j)$ such that it is crucial to send the momenta to zero only at the end.

There are, in principle, several terms stemming from whether one takes the bosonic or fermionic parts of the gauge fields. First of all, one can show that the terms needed to reinforce BRST invariance of the gauge fields vertex operators do not lead to any non-trivial contributions. There are thus four separate CFT correlators all of which are multiplied by

$$A_0 = \frac{1}{2^9 \pi^2} \epsilon^{AB} \epsilon^{\hat{A}\hat{B}} A_\mu A_\rho (\sigma^{\lambda\kappa})^{\alpha\beta} F_{\lambda\kappa} (\sigma^{\omega\xi})^{\gamma\delta} \bar{F}_{\omega\xi} \frac{z_{12} z_{14} z_{24}}{(z_{34} z_{35} z_{36} z_{45} z_{46} z_{56})^{\frac{1}{4}}} \prod_{\substack{i,j=1 \\ i < j}}^6 z_{ij}^{4p_i \cdot p_j}. \quad (3.31)$$

Here, I have defined $p_3 = p_4 = P/2$ and $p_5 = p_6 = \bar{P}/2$. If one takes both gauge fields to give their bosonic term, one obtains the term

$$\begin{aligned} A_1 &= \langle \partial X^\mu(z_2) \partial X^\rho(z_2) \rangle \langle S_\alpha S_A(z_3) S_\beta S_B(z_4) S_\gamma S_{\hat{A}}(z_5) S_\delta S_{\hat{B}}(z_6) \rangle \\ &= -\frac{\epsilon_{AB} \epsilon_{\hat{A}\hat{B}} \epsilon_{\alpha\delta} \epsilon_{\beta\gamma} z_{36} z_{45}}{(z_{34} z_{35} z_{36} z_{45} z_{46} z_{56})^{\frac{3}{4}}} \left(2 \frac{p_2^\mu}{z_{12}} + \frac{P^\mu}{z_{13}} + \frac{P^\mu}{z_{14}} + \frac{\bar{P}^\mu}{z_{15}} + \frac{\bar{P}^\mu}{z_{16}} \right) \\ &\quad \times \left(2 \frac{p_1^\rho}{z_{12}} - \frac{P^\rho}{z_{23}} - \frac{P^\rho}{z_{24}} - \frac{\bar{P}^\rho}{z_{25}} - \frac{\bar{P}^\rho}{z_{26}} \right). \end{aligned} \quad (3.32)$$

Notice that I have only kept the terms that would be non vanishing once I take into account the polarisation vectors in A_0 . However, in the gauge chosen for the worldsheet positions, this term is zero by the transversality condition $p_1^\mu A_\mu = 0$. The second term obtained when only the gauge field at z_2 gives its bosonic piece is

$$\begin{aligned}
 A_2 &= 2ip_{1,v} \langle \partial X^\rho(z_2) \rangle \langle \psi^\mu \psi^\nu(z_1) S_\alpha S_A(z_3) S_\beta S_B(z_4) S_\gamma S_{\hat{A}}(z_5) S_\delta S_{\hat{B}}(z_6) \rangle \\
 &= \frac{-P_{1,v} \in_{AB} \in_{\hat{A}\hat{B}} z_{34} z_{56}}{z_{13} z_{14} z_{15} z_{16} (z_{34} z_{35} z_{36} z_{45} z_{46} z_{56})^{\frac{3}{4}}} \left[\epsilon_{\beta\gamma} (\sigma^{\mu\nu})_{\alpha\delta} z_{14} z_{15} z_{36} + \epsilon_{\alpha\delta} (\sigma^{\mu\nu})_{\beta\gamma} z_{13} z_{16} z_{45} \right] \\
 &\times \left(2 \frac{p_1^\rho}{z_{12}} - \frac{P^\rho}{z_{23}} - \frac{P^\rho}{z_{24}} - \frac{\bar{P}^\rho}{z_{25}} - \frac{\bar{P}^\rho}{z_{26}} \right) \quad (3.33)
 \end{aligned}$$

Similarly to A_1 , when the gauge field at z_1 only gives its bosonic piece then the correlator vanishes. Finally, when all vertices give their fermionic terms, one finds

$$\begin{aligned}
 A_4 &= -4p_{1,v} p_{2,\sigma} \langle \psi^\mu \psi^\nu(z_1) \psi^\rho \psi^\sigma(z_2) S_\alpha S_A(z_3) S_\beta S_B(z_4) S_\gamma S_{\hat{A}}(z_5) S_\delta S_{\hat{B}}(z_6) \rangle \\
 &= \frac{2p_{1,v} p_{2,\sigma} \in_{AB} \in_{\hat{A}\hat{B}}}{z_{12} z_{13} z_{14} z_{15} z_{16} z_{23} z_{24} z_{25} z_{26} (z_{34} z_{35} z_{36} z_{45} z_{46} z_{56})^{\frac{3}{4}}} \\
 &\times \left[\delta^{\mu\rho} \epsilon_{\beta\gamma} (\sigma^{\nu\bar{\sigma}})_{\alpha\delta} z_{13} z_{15}^2 z_{24}^2 z_{26} z_{34} z_{36} z_{56} \right. \\
 &+ \delta^{\mu\rho} \epsilon_{\alpha\delta} (\sigma^{\nu\bar{\sigma}})_{\gamma\beta} z_{13}^2 z_{15} z_{24} z_{26}^2 z_{34} z_{45} z_{56} \\
 &+ \delta^{\nu\sigma} \epsilon_{\beta\gamma} (\sigma^{\mu\bar{\sigma}})_{\alpha\delta} z_{13} z_{15}^2 z_{24}^2 z_{26} z_{34} z_{36} z_{56} \\
 &+ \delta^{\nu\sigma} \epsilon_{\alpha\delta} (\sigma^{\mu\bar{\sigma}})_{\gamma\beta} z_{13}^2 z_{15} z_{24} z_{26}^2 z_{34} z_{45} z_{56} \\
 &- \delta^{\mu\sigma} \epsilon_{\beta\gamma} (\sigma^{\nu\bar{\sigma}})_{\alpha\delta} z_{13} z_{15}^2 z_{24}^2 z_{26} z_{34} z_{36} z_{56} \\
 &- \delta^{\mu\sigma} \epsilon_{\alpha\delta} (\sigma^{\nu\bar{\sigma}})_{\gamma\beta} z_{13}^2 z_{15} z_{24} z_{26}^2 z_{34} z_{45} z_{56} \\
 &- \delta^{\nu\rho} \epsilon_{\beta\gamma} (\sigma^{\mu\bar{\sigma}})_{\alpha\delta} z_{13} z_{15}^2 z_{24}^2 z_{26} z_{34} z_{36} z_{56} \\
 &- \delta^{\nu\rho} \epsilon_{\alpha\delta} (\sigma^{\mu\bar{\sigma}})_{\gamma\beta} z_{13}^2 z_{15} z_{24} z_{26}^2 z_{34} z_{45} z_{56} \\
 &- \frac{1}{2} (\sigma^{\mu\bar{\sigma}} \sigma^{\nu\bar{\sigma}} \sigma^{\rho\bar{\sigma}})_{\alpha\delta} \epsilon_{\beta\gamma} z_{12} z_{14} z_{15} z_{24} z_{25} z_{34} z_{36}^2 z_{56} \\
 &+ \frac{1}{2} (\sigma^{\mu\bar{\sigma}} \sigma^{\nu\bar{\sigma}} \sigma^{\rho\bar{\sigma}})_{\gamma\beta} \epsilon_{\alpha\delta} z_{12} z_{13} z_{16} z_{23} z_{26} z_{34} z_{45}^2 z_{56} \\
 &+ \frac{1}{2} z_{12} \left((\sigma^{\mu\bar{\sigma}})_{\alpha\delta} (\sigma^{\rho\bar{\sigma}})_{\gamma\beta} z_{12} z_{15} z_{26}^2 z_{34} z_{36} z_{45} z_{56} \right. \\
 &+ (\sigma^{\mu\bar{\sigma}})_{\gamma\beta} (\sigma^{\rho\bar{\sigma}})_{\alpha\delta} z_{12} z_{13} z_{24} z_{34} z_{36} z_{45} z_{56}^2 \Big) \\
 &\left. - \frac{1}{2} z_{12} z_{13} z_{15} z_{24} z_{26} z_{34} z_{36} z_{45} z_{56} \left((\sigma^{\nu\bar{\sigma}})_{\alpha\delta} (\sigma^{\mu\bar{\sigma}})_{\gamma\beta} + (\sigma^{\nu\bar{\sigma}})_{\gamma\beta} (\sigma^{\mu\bar{\sigma}})_{\alpha\delta} \right) \right] \quad (3.34)
 \end{aligned}$$

Here, I have used the explicit results of [17] for higher point fermionic correlation function on the disc. This can be further simplified to

$$\begin{aligned}
A_4 = & \frac{2p_{1,v} p_{2,\sigma} \epsilon_{AB} \epsilon_{\hat{A}\hat{B}}}{z_{12} z_{13} z_{14} z_{15} z_{16} z_{23} z_{24} z_{25} z_{26} (z_{34} z_{35} z_{36} z_{45} z_{46} z_{56})^{\frac{3}{4}}} \left\{ \right. \\
& + z_{13} z_{15}^2 z_{24}^2 z_{26} z_{34} z_{36} z_{56} (\delta^{\mu\rho} \epsilon_{\beta\gamma} (\sigma^\nu \bar{\sigma}^\sigma)_{\alpha\delta} + \delta^{\nu\sigma} \epsilon_{\beta\gamma} (\sigma^\mu \bar{\sigma}^\rho)_{\alpha\delta} \\
& - \delta^{\mu\sigma} \epsilon_{\beta\gamma} (\sigma^\nu \bar{\sigma}^\rho)_{\alpha\delta} - \delta^{\nu\rho} \epsilon_{\beta\gamma} (\sigma^\mu \bar{\sigma}^\sigma)_{\alpha\delta}) \\
& + z_{13}^2 z_{15} z_{24} z_{26}^2 z_{34} z_{45} z_{56} (\delta^{\mu\rho} \epsilon_{\alpha\delta} (\sigma^\nu \bar{\sigma}^\sigma)_{\gamma\beta} + \delta^{\nu\sigma} \epsilon_{\alpha\delta} (\sigma^\mu \bar{\sigma}^\rho)_{\gamma\beta} \\
& - \delta^{\mu\sigma} \epsilon_{\alpha\delta} (\sigma^\nu \bar{\sigma}^\rho)_{\gamma\beta} - \delta^{\nu\rho} \epsilon_{\alpha\delta} (\sigma^\mu \bar{\sigma}^\sigma)_{\gamma\beta}) \\
& - \frac{1}{2} (\sigma^\mu \bar{\sigma}^\nu \sigma^\rho \bar{\sigma}^\sigma)_{\alpha\delta} \epsilon_{\beta\gamma} z_{12} z_{14} z_{15} z_{24} z_{25} z_{34} z_{36}^2 z_{56} \\
& + \frac{1}{2} (\sigma^\mu \bar{\sigma}^\nu \sigma^\rho \bar{\sigma}^\sigma)_{\gamma\beta} \epsilon_{\alpha\delta} z_{12} z_{13} z_{16} z_{23} z_{26} z_{34} z_{45}^2 z_{56} \\
& + \frac{1}{2} z_{12} z_{34} z_{36} z_{45} z_{56} \left[z_{12} z_{15} z_{26} z_{34} (\sigma^\mu \bar{\sigma}^\nu)_{\alpha\delta} (\sigma^\rho \bar{\sigma}^\sigma)_{\gamma\beta} \right. \\
& + z_{12} z_{13} z_{24} z_{56} (\sigma^\mu \bar{\sigma}^\nu)_{\gamma\beta} (\sigma^\rho \bar{\sigma}^\sigma)_{\alpha\delta} \\
& \left. - z_{13} z_{15} z_{24} z_{26} ((\sigma^\nu \bar{\sigma}^\rho)_{\alpha\delta} (\sigma^\mu \bar{\sigma}^\sigma)_{\gamma\beta} + (\sigma^\nu \bar{\sigma}^\rho)_{\gamma\beta} (\sigma^\mu \bar{\sigma}^\sigma)_{\alpha\delta}) \right] \left. \right\}. \quad (3.35)
\end{aligned}$$

The full correlator is thus $A_0(A_2 + A_4)$ which I now integrate over the worldsheet positions x , z_5 and z_6 . For generic space-time momenta, this is a well defined integral over $\mathbb{R} \times \mathbb{C}$. Instead of performing directly the calculation using the analytic structure of the integrand, I use the beautiful results of [18] in which this type of complex integrals is mapped to real integrals appearing in colour ordered amplitudes in gauge theory. More precisely, it was proven that

$$\begin{aligned}
& \int_{\mathbb{R} \times \mathbb{C}} dx d^2z (z - \bar{z})^\kappa x^\alpha (1 + ix)^{u+n_0} (1 - ix)^{u+n_1} (1 - z)^{t+n_2} (1 - \bar{z})^{t+n_3} (z - ix)^{\frac{s}{2}+n_4} \\
& \quad \times (\bar{z} + ix)^{\frac{s}{2}+n_5} (z + ix)^{\frac{s}{2}+n_6} (\bar{z} - ix)^{\frac{s}{2}+n_7} \\
& = 2 \sin\left(\frac{\pi s}{2}\right) \sin(\pi s) A(1, 6, 3, 5, 4, 2) - 2 \sin\left(\frac{\pi s}{2}\right) \sin(\pi t) A(1, 3, 5, 4, 2, 6), \quad (3.36)
\end{aligned}$$

with

$$\begin{aligned}
& A(1, 6, 3, 5, 4, 2) \\
& = -\frac{1}{2} (-)^{n_4+n_5+n_7} \int_0^1 dx x^{u-2-\kappa-n_3-n_5-n_7} (1-x)^{\frac{s}{2}+n_5} \int_0^1 dy y^{-\frac{s}{2}+n_1+n_2+n_4+1} \\
& \quad \times (1-y)^{\frac{s}{2}+n_6} (1-xy)^\kappa \int_0^1 dz z^{u+n_1} (1-z)^{\frac{s}{2}+n_4} (1-yz)^\alpha (1-xyz)^{\frac{s}{2}+n_7}, \quad (3.37)
\end{aligned}$$

and

$$\begin{aligned}
& A(1, 3, 5, 4, 2, 6) \\
& = \frac{1}{2} (-)^{\kappa+n_3+n_4} \int_0^1 dx x^{\frac{s}{2}+n_6} (1-x)^{\frac{s}{2}+n_4} \int_0^1 dy y^{s+\alpha+n_4+n_6+1} (1-y)^{u+n_1} (1-xy)^{t+n_1}
\end{aligned}$$

$$\times \int_0^1 dz z^{u-\kappa-n_3-n_5-n_7-2} (1-z)^{t+n_3} (1-yz)^{\frac{s}{2}+n_6}. \quad (3.38)$$

These integrals are generically quite involved. However, recall that in the present case, I am only interested in the leading contribution in the space-time momenta. The latter stems from contact terms arising from the worldsheet integrals, *i.e.* poles in s , t or u where

$$\begin{aligned} s &= 2p_1 \cdot p_2 = 2P \cdot \bar{P}, \\ s &= 2p_1 \cdot P = 2p_2 \cdot \bar{P}, \\ s &= 2p_1 \cdot \bar{P} = 2p_2 \cdot P. \end{aligned} \quad (3.39)$$

Once the leading contributions from the worldsheet integrals is extracted, one evaluates the total amplitude $A_0(A_2 + A_4)$ in the limit of vanishing momenta. The identities for the traces of Pauli matrices derived in Appendix A turn out to be very useful. The result takes the following simple form:

$$\left\langle \left\langle V_A(z_1) V_A(z_2) V_G^{(-\frac{1}{2}, -\frac{1}{2})}(z_3, z_4) V_{S'}^{(-\frac{1}{2}, -\frac{1}{2})}(z_5, z_6) \right\rangle \right\rangle = A_\mu A^\mu F_{G, \rho\sigma} F_{S'}^{\rho\sigma}. \quad (3.40)$$

One can perform the same analysis for all other possible terms, for instance when including the NS-NS parts of the vertex operators, and I find

$$\langle V_A V_A V_G V_{S'} \rangle = \epsilon \bar{\epsilon} A_\mu A^\mu. \quad (3.41)$$

Before concluding this session, one should check that the full deformation does not completely break supersymmetry, even in the presence of $\bar{\Omega}$. More precisely, this boils down to showing that the scalar component of the gauge multiplet remains massless. In order to achieve that, one should calculate all possible couplings between two scalars ϕ , $\bar{\phi}$ and the Ω , $\bar{\Omega}$ deformation in the field theory limit. Instead of performing the disc amplitude calculations as before, I simply note that this precise analysis has already been done in [19] where it was shown that the mass deformation realised at the level of the worldsheet does not lead to a mass term for the Higgs scalar. The situation here is exactly the same by a mere exchange of the space-time \mathbb{C}^2 and the \mathbb{C}^2 of the internal space used in [19]. It is important to note that the role of the quadratic deformation is crucial to ensure this property. Namely, as extensively studied in [1], the full description of the Ω -deformation requires the introduction of a background for the scalar field S' at second order in momenta and proportional to $\epsilon \bar{\epsilon}$. As reviewed in Section 4, in heterotic string theory, this additional ingredient implies an exact decoupling of $\bar{\Omega}$ from the topological amplitudes F_g as a cancellation between bosonic and fermionic degrees of freedom. In the present case, it ensures that supersymmetry is not broken while it does not change the form of the remaining couplings.

To summarise our findings, I write down the full Ω -deformed Yang-Mills action derived from the realisation of the deformation as a constant background of particular string states. The action is

$$\begin{aligned} \mathcal{L}_{\text{SYM}} = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left\{ \frac{1}{2} F^2 - 2 \bar{\Lambda}_{\dot{\alpha}A} \left(\bar{\mathcal{D}}^{\dot{\alpha}\beta} \Lambda_{\beta}^A + i\sqrt{2} \left[\phi, \bar{\Lambda}^{\dot{\alpha}A} \right] \right) \right. \\ \left. + 2i\sqrt{2} \Lambda^{\alpha}{}_A \left(\epsilon_{\alpha\beta} [\bar{\phi}, \Lambda_{\beta}^A] + \frac{\bar{\epsilon}}{2} (\sigma^3)_{\alpha}^{\beta} \Lambda_{\beta}^A \right) \right\} \end{aligned}$$

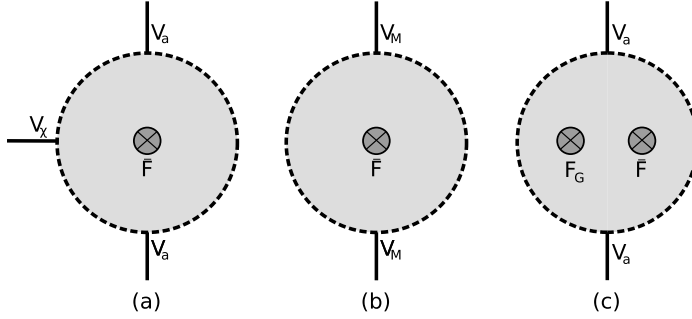


Fig. 2. Disc diagrams in the 5-5 sector. Diagram (a) involves three bosonic boundary insertions and one bulk insertion of \bar{F} , diagram (b) two fermionic insertions and one \bar{F} , whereas diagram (c) has two bosonic boundary insertions and two bulk insertions of a graviphoton and an \bar{F} .

$$+ \left(D_\mu \phi - \epsilon A^\nu \eta_{\mu\nu}^3 \right) \left(D^\mu \bar{\phi} - \bar{\epsilon} A_\rho \eta^{3,\mu\rho} \right) \Big\} + \mathcal{O}(\alpha'), \quad (3.42)$$

which shows that the realisation through the graviphoton and the S' -vector is consistent. In the following section, I prove that this statement holds non-perturbatively.

3.3. Deformed ADHM action

I now focus on the deformation of the ADHM action stemming from the closed string vertices. As I now show, this confirms the choice of the vertex made previously and leads to the explicit expression of the $\bar{\Omega}$ -deformed instanton effective action as a check. As in the Yang-Mills case, there is a limited number of couplings surviving the field theory limit and self-duality projections. Some of these are summarised in Fig. 2.

In principle, there could as well be couplings with mixed D5/D9 boundary conditions. However, due to the self-duality constraint, such couplings are zero. In the case of a general Ω -background, this is of course not any longer true [8].

I first consider the coupling involving fermionic ADHM moduli. That is, for the NS-NS part,

$$\begin{aligned} & \left\langle \left\langle c V_M(z_1) c V_M(z_2) c V_{\bar{F}}^{(-1,0)}(z_3, z_4) \right\rangle \right\rangle \\ &= \frac{1}{8\pi\sqrt{2}} M^{\alpha A} M^{\beta B} \bar{F}_{\mu\nu} \left\langle \left\langle c(z_1) S_\alpha(z_1) S_A(z_1) e^{-\frac{1}{2}\varphi(z_1)} \right. \right. \\ & \quad \times c(z_2) S_\beta(z_2) S_B(z_2) e^{-\frac{1}{2}\varphi(z_2)} \\ & \quad \left. \left. \times c(z_3) e^{-\varphi(z_3)} \bar{\Psi}(z_3) \psi^\mu \psi^\nu(z_4) \right\rangle \right\rangle + (\text{Left} \leftrightarrow \text{Right}). \end{aligned} \quad (3.43)$$

Notice that, apart from the fact that space-time momenta are forbidden for the ADHM moduli due to the Dirichlet boundary conditions, the calculation goes along the same lines as (3.10). In addition, the g_6^2 factor from the vertices (3.4) cancels with the normalisation of the D5 amplitude. Therefore, I can immediately state the result. Namely, for the S one has

$$\left\langle V_M V_M V_S \right\rangle = 0 \quad (3.44)$$

while

$$\langle V_M V_M V_{S'} \rangle = -i\sqrt{2}\bar{\epsilon} M^{\alpha A} M^{\beta}{}_A (\sigma^3)_{\alpha\beta}. \quad (3.45)$$

I now turn to the bosonic term. First consider the coupling corresponding to Diagram (a) in Fig. 2. Once again, for simplicity, I choose the zero-picture vertices to be of dimension one (such that their positions are integrated), and all the (-1) -picture vertices to be of dimension zero (such that their positions remain unintegrated) so that one must insert one PCO. As I show below, the amplitude takes the form of contact terms in the momenta of the form $p_i \cdot p_j / p_i \cdot p_j$. These contact terms survive in the limit $p_i \rightarrow 0$. To be able to compute them in a well-defined manner, the momenta p_i must be kept generic as they also act as regulators of the worldsheet integrals. The limit of vanishing momenta is only taken at the end of the calculation. Notice, however, that due to the nature of our vertex insertions, none of the ADHM moduli can carry momenta along X^μ since the four-dimensional space-time corresponds to directions with Dirichlet boundary conditions for the D5-instantons. Similarly, the S' -vector insertions cannot carry momenta along T^2 because of BRST invariance. As a way out, I turn on complex momenta along the $K3$ directions for all the vertices to make all integrals well-defined. Technically, this means that I first replace $K3$ by \mathbb{R}^4 and compute the amplitude on the D-instanton world-volume $T^2 \times \mathbb{R}^4$. Since the fields appearing in this amplitude survive the orbifold projection,² the coupling of interest also exists in the case where \mathbb{R}^4 is replaced by $K3$.

As the role of space-time and internal momenta is crucial, and for the sake of clarity, I write the relevant vertex operators below.

$$V_a(z_1) = g_6 a_\mu (\partial X^\mu - 2ip_1 \cdot \chi \psi^\mu) e^{2ip_1 \cdot Y}(z_1), \quad (3.46)$$

$$V_a(z_2) = g_6 a_\nu c e^{-\varphi} \psi^\nu e^{2ip_2 \cdot Y}(z_2), \quad (3.47)$$

$$V_\chi(z_3) = \frac{\chi}{\sqrt{2}} (\partial Z - 2ip_3 \cdot \chi \Psi) e^{2ip_3 \cdot Y}(z_3), \quad (3.48)$$

$$V_{\bar{F}}(z) = c e^{-\varphi} \bar{\Psi} e^{i(P_\mu X^\mu + P \cdot Y)}(z), \quad (3.49)$$

$$V_{\bar{F}}(\bar{z}) = -\frac{i\epsilon_\lambda}{8\pi\sqrt{2}} c e^{-\varphi} \psi^\lambda e^{i(-P_\mu X^\mu + P \cdot Y)}(\bar{z}), \quad (3.50)$$

with $\bar{F}_{\mu\nu} \equiv \epsilon_{[\mu} P_{\nu]}$, and the only relevant terms in V_{PCO} are

$$e^\varphi T_F(y) = i e^\varphi (\psi^\mu \partial X_\mu + \Psi \partial \bar{Z} + \bar{\Psi} \partial Z + \chi^i \partial Y^i)(y). \quad (3.51)$$

Here, $Y^i \in \{X^6, X^7, X^8, X^9\}$ are coordinates of the internal \mathbb{R}^4 . The momenta p_i are along these directions, while the momentum of $V_{FS'}$ is written as (P_μ, P) , where P_μ is the space-time part and P is along the Y^i directions. Note that after using the doubling trick, the Neumann directions (Z, \bar{Z}, Y^i) are mapped onto themselves, whereas the Dirichlet ones pick an additional minus sign $X^\mu \rightarrow -X^\mu$. This is consistent with the fact that the momenta along Neumann directions are conserved. On the other hand, integrating over the zero modes of the Dirichlet directions X^μ does not give rise to any conservation law for the momentum P_μ .

The three open string vertices contain Chan-Paton labels that should be suitably ordered. For instance, in order to obtain the term $\text{Tr}(a_\mu a_\nu \chi)$, the range of integration is

$$\begin{cases} \text{for } z_1 > z_2, & z_3 \in]z_2, z_1[, \\ \text{for } z_2 > z_1, & z_3 \in]-\infty, z_1[\cup]z_2, \infty[. \end{cases} \quad (3.52)$$

² More generally on a smooth $K3$ manifold they give rise to zero-modes.

For the other nonequivalent ordering $\text{Tr}(a_\mu \chi a_\nu)$, the range of integration in z_3 is opposite. It is easy to show that the sum of these two orderings gives $\text{Tr}(a_\mu [\chi, a_\nu])$.

For definiteness, I first focus on the term $\text{Tr}(a_\mu a_\nu \chi)$. The contractions of the ghosts, superghosts and the exponentials in momenta yield

$$A_0 = \left\{ -\frac{ig_6^2}{16\pi} \text{Tr}(a_\mu a_\nu \chi) \epsilon_\lambda \right\} |y-z|^2 (y-z_2) \\ \times \prod_{1 \leq i < j \leq 3} (z_{ij})^{4p_i \cdot p_j} \prod_{k=1}^3 |z_k - z|^{4p_k \cdot P} (z - \bar{z})^{-P_\mu P^\mu + P_i P_i}. \quad (3.53)$$

This factor multiplies each of the remaining contractions. Now consider the contribution of $\partial Z(z_3)$ to the amplitude. This can only contract with $\partial \bar{Z}(y)$ in V_{PCO} while $\Psi(y)$ contracts with $\bar{\Psi}(z)$. Hence, $\psi^\lambda(\bar{z})$ must contract with $\psi^\nu(z_2)$ and from z_1 only $\partial X^\mu(z_1)$ can contribute. The result is

$$A_1 = \frac{i\delta^{\nu\lambda} P^\mu (z - \bar{z})}{(y - z_3)^2 (y - z)(z_2 - \bar{z})|z_1 - z|^2}. \quad (3.54)$$

Next consider the contribution of the second term in (3.48) where there are several contributions. First, if $p_3 \cdot \chi(z_3)$ contracts with $p_1 \cdot \chi(z_1)$, then $\psi^\mu(z_1)$, $\psi^\nu(z_2)$, $\psi^\lambda(\bar{z})$ and a space-time fermion $\psi^\sigma(y)$ from the PCO must contract, leaving $\partial X^\sigma(y)$ to contract with the momentum parts of the operators at z and \bar{z} . This gives a term proportional to P_σ . Secondly, notice that the term arising from contracting ψ^μ with ψ^ν is killed by the transversality condition. The total result is

$$A_2 = \frac{4ip_1 \cdot p_3 (z - \bar{z})}{(z_3 - z)z_3|y - z|^2} \left[\frac{\delta^{\nu\lambda} P^\mu}{(z_2 - \bar{z})(y - x_1)} - \frac{\delta^{\mu\lambda} P^\nu}{(z_1 - \bar{z})(y - z_2)} \right]. \quad (3.55)$$

On the other hand, if the term $p_3 \cdot \chi(z_3)$ contracts with $\chi(y)$ in V_{PCO} , $\partial Y(y)$ must contract with momentum dependent parts of the vertices. Thus, $\psi^\lambda(\bar{z})$ contracts with $\psi^\nu(z_2)$ and only ∂X^μ at z_1 contributes so that one obtains

$$A_3 = \frac{4i\delta^{\nu\lambda} P^\mu (z - \bar{z})}{(z_3 - z)(y - z_3)(z_2 - \bar{z})|z_1 - z|^2} \left[\frac{p_3 \cdot p_1}{y - z_1} + \frac{p_3 \cdot p_2}{y - z_2} + \frac{p_3 \cdot P}{2(y - z)} + \frac{p_3 \cdot \bar{P}}{2(y - \bar{z})} \right]. \quad (3.56)$$

The total correlation function is thus given by

$$\left\langle V_a(z_1) V_a(z_2) V_\chi(z_3) V_{\bar{F}}^{\text{NS-NS}} \right\rangle = A_0(A_1 + A_2 + A_3), \quad (3.57)$$

which I now integrate over z_1 and z_3 . Note that all the terms in A_1 , A_2 and A_3 come with a single power of space-time momentum P^μ which is exactly what is required to obtain a coupling to the field strength of the closed string state. However, both A_2 and A_3 are quadratic in the momenta along the Y^i directions. Hence, they can only contribute to the amplitude in the zero-momentum limit if the integration over z_1 and z_3 gives a pole $1/(p_a \cdot p_b)$. Clearly, $A_0 \cdot A_3$ cannot give such a pole.³ On the other hand, the integral over z_3 for $A_0 \cdot A_2$ gives a $1/(p_1 \cdot p_3)$ pole. Performing

³ I am assuming a generic value of y in the complex plane.

the z_3 integral in the regions (3.52) yields the same result. Consequently, the z_1 integral over the entire real line reads

$$A_0 A_2 = \frac{g_6^2}{8\pi} \text{Tr} [a_\mu a_\nu \chi] \int_{-\infty}^{\infty} dz_1 \frac{(z - \bar{z})}{|z_1 - z|^2} \epsilon_\lambda \left[\frac{(z_1 - \bar{z})(y - z_2)}{(z_2 - \bar{z})(y - z_1)} P^\mu \delta^{\nu\lambda} - P^\nu \delta^{\mu\lambda} \right], \quad (3.58)$$

where I have set all the momenta along the Y^i directions to zero given that there are no singularities in the remaining integral. Note that the integral $A_0 \cdot A_2$ is not gauge invariant. As for the $A_0 A_1$ term the z_3 and z_1 integrals contain no singularities so that the momenta along the Y^i directions can be set to zero. The resulting z_3 integral for both regions (3.52) gives the same result:

$$A_0 A_1 = -\frac{g_6^2}{8\pi} \text{Tr} [a_\mu a_\nu \chi] \int_{-\infty}^{\infty} dx_1 \frac{(z - \bar{z})}{|z_1 - z|^2} \epsilon_\lambda \frac{(y - \bar{z})z_{12}}{(z_2 - \bar{z})(y - z_1)} P^\mu \delta^{\nu\lambda}. \quad (3.59)$$

Adding the two terms (3.58) and (3.59), the total result becomes gauge invariant and the z_1 integration yields⁴:

$$\left\langle \left\langle V_a V_a V_\chi V_{\bar{F}}^{\text{NS-NS}} \right\rangle \right\rangle = -2i \text{Tr} [a_\mu a_\nu \chi] \epsilon_\lambda (P^\mu \delta^{\nu\lambda} - P^\nu \delta^{\mu\lambda}). \quad (3.60)$$

Finally, let us consider the R-R contributions. The vertex operators are the same as above, except for the closed string vertex:

$$V_{\bar{F}}^{(-\frac{1}{2}, -\frac{1}{2})}(z, \bar{z}) = \frac{i}{16\pi\sqrt{2}} \bar{F}^{\rho\lambda} c e^{-\frac{\varphi}{2}} S_\alpha S_{\hat{A}} e^{i(P \cdot X + P \cdot Y)}(z) \\ \times \epsilon^{\hat{A}\hat{B}} (\sigma_{\rho\lambda})^\alpha_\beta c e^{-\frac{\varphi}{2}} S^\beta S_{\hat{B}} e^{i(-P \cdot X + P \cdot Y)}(\bar{z}). \quad (3.61)$$

Since the total superghost charge is -2 , there is no need for a PCO. The total T^2 charge implies that only $p_3 \cdot \chi \Psi(x_3)$ in (3.48) contributes and, thus, only $p_1 \cdot \chi \psi^\mu(x_1)$ in (3.46) contributes. This term is proportional to $p_1 \cdot p_3$. As before, the integral over z_3 gives a pole $1/(p_1 \cdot p_3)$ in the channel where $z_3 \rightarrow z_1$. Performing the integrals over z_1 and z_3 gives the same result:

$$\left\langle \left\langle V_a V_a V_\chi V_{\bar{F}}^{\text{NS-NS}} \right\rangle \right\rangle = \left\langle \left\langle V_a V_a V_\chi V_{\bar{F}}^{\text{R-R}} \right\rangle \right\rangle. \quad (3.62)$$

Finally, summing over the nonequivalent orderings of the open vertex operators gives

$$\left\langle \left\langle V_a V_a V_\chi V_{\bar{F}} \right\rangle \right\rangle = -4i \text{Tr} [\chi, a_\mu] a_\nu F_{S'}^{\mu\nu}, \quad (3.63)$$

while for V_S it is zero as before!

To complete the analysis, I now calculate the coupling involving both Ω and $\bar{\Omega}$. This is the exact counterpart of (3.41) for the instanton sector. Thus, the calculation goes along the same lines as the one leading to (3.41), with the subtle difference that, in the present case, momenta can only be turned on in the $K3$ directions. Consequently, I obtain a non-trivial quadratic coupling with Ω and $\bar{\Omega}$ making the instanton modulus a massive, and the full deformation of the ADHM action including both self-dual Ω and $\bar{\Omega}$ is

⁴ I included an additional factor of 2 stemming from the left-right symmetrisation in the closed string vertex.

$$S_{\text{ADHM}} = -\text{Tr} \left\{ \left([\chi^\dagger, a_{\alpha\dot{\beta}}] + \bar{\epsilon}(\sigma_3 a)_{\alpha\dot{\beta}} \right) \left([\chi, a^{\dot{\beta}\alpha}] + \epsilon(a\sigma_3)^{\dot{\beta}\alpha} \right) \right. \\ \left. + \frac{1}{2} M^{\alpha A} \left([\chi^\dagger, M_{\alpha A}] + \bar{\epsilon}(\sigma_3)_{\alpha}{}^{\beta} M_{\beta A} \right) \right\}. \quad (3.64)$$

Clearly, I have only displayed the $\Omega, \bar{\Omega}$ dependent terms.

4. $\bar{\Omega}$ and topological amplitudes

4.1. Effective action

I now briefly review the results of [1] for completeness. There, based on the fact that parts of the topological amplitudes F_g can be computed perturbatively in heterotic at one-loop using the couplings [6]

$$\mathcal{I}_g = \int d^4x \int d^4\theta \mathcal{F}_g(X) (W_{\mu\nu}^{ij} W_{ij}^{\mu\nu})^g \quad \text{for} \quad g \geq 1, \quad (4.1)$$

we proposed a deformation of the latter to capture also $\bar{\Omega}$ as a constant field strength for the vector partner of the Kähler modulus of T^2 denoted F_T .⁵ Recall that $W_{\mu\nu}^{ij}$ is the supergravity multiplet carrying (anti-symmetrised) indices $i, j = 1, 2$ for the $SU(2)_R$ R-symmetry group. It contains the graviphoton field-strength F^G , the field strength tensor $B_{\mu\nu}^i$ of an $SU(2)$ doublet of gravitini and the Riemann tensor.

In order to include additional insertions of F_T , a naive ansatz is to consider the deformed coupling

$$\mathcal{I}_{g,n} = \int d^4x \int d^4\theta \mathcal{F}_{g,n}(X) W^{2g} K_T^{2n}, \quad (4.2)$$

where K_T is the descendent superfield whose lowest component is F_T . However, this coupling not only breaks supersymmetry, but also does not encode the quadratic terms that are crucial in order to obtain a good description of $\bar{\Omega}$ [1]. More precisely, one should also include an arbitrary number of the scalar field T at two derivatives. At the level of the worldsheet sigma model, this translates into a quadratic deformation proportional to $\epsilon\bar{\epsilon}$ as in type I. It would be interesting to construct an effective coupling that includes the latter.

4.2. Amplitude calculation

The couplings $F_{g,n}$ are naturally calculated perturbatively at one-loop.⁶ For convenience, I choose particular kinematics in which the states of interest carry space-time momenta along Z^1 and \bar{Z}^2 only. The vertex operators for the graviphotons are

$$V_G(z_i) = (\partial X - i\bar{p}_1\psi^2\Psi)\bar{\partial}Z^2 e^{i\bar{p}_1Z^1}, \\ V_G(w_i) = (\partial X - ip_2\bar{\psi}^2\Psi)\bar{\partial}\bar{Z}^1 e^{ip_2\bar{Z}^2}, \quad (4.3)$$

while for the T -vectors these are

⁵ Recall that the dual heterotic theory is also compactified on $T^2 \times K3$.

⁶ Notice the different notation compared to the couplings appearing in (4.2), i.e. $F_{g,n}$ is not calculated by the latter.

$$\begin{aligned}
V_T(s_i) &= (\partial Z^2 - i \bar{p}_1 \psi^1 \psi^2) \bar{\partial} \bar{X} e^{i \bar{p}_1 Z^1}, \\
V_T(t_i) &= (\partial \bar{Z}^2 - i p_2 \bar{\psi}^2 \bar{\psi}^1) \bar{\partial} \bar{X} e^{i p_2 \bar{Z}^2}.
\end{aligned} \tag{4.4}$$

For simplicity, I consider the coupling of $2g$ graviphotons and $2n + 2$ field strengths V_T so that the space-time zero-modes are soaked up by two vertices V_T . Hence, the amplitude to calculate is

$$\left\langle V_T(x) V_T(y) \prod_{i=1}^g V_G(z_i) V_G(w_i) \prod_{j=1}^n V_T(s_j) V_T(t_j) \prod_{k=1}^m V_\phi(u_k) \right\rangle, \tag{4.5}$$

which corresponds to the second derivative of $F_{g,n}$ with respect to T [1]. Here, V_ϕ is the vertex operators of the scalar field T . After soaking up the space-time zero-modes, one finds that the coupling is integrable in the sense that one can pull out the two derivatives with respect to T . Furthermore, by summing over g , n and m , one can define a generating function in which all the amplitudes with arbitrary number of fields reduce to a Gaussian deformation of the worldsheet sigma-model! That is, the generating function

$$\begin{aligned}
F(\epsilon, \bar{\epsilon}) &= \left\langle \exp \left[-\epsilon \int d^2 z \left(\partial X Z^1 \bar{\partial} Z^2 + \partial X \bar{Z}^2 \bar{\partial} \bar{Z}^1 \right) \right. \right. \\
&\quad \left. \left. - \bar{\epsilon} \int d^2 z \left((Z^1 \partial Z^2 + \psi^1 \psi^2) \bar{\partial} \bar{X} + (\bar{Z}^2 \partial \bar{Z}^1 + \bar{\psi}^1 \bar{\psi}^2) \bar{\partial} \bar{X} \right) \right. \right. \\
&\quad \left. \left. + \epsilon \bar{\epsilon} \int d^2 z |Z^i|^2 \partial X \bar{\partial} \bar{X} \right] \right\rangle,
\end{aligned} \tag{4.6}$$

calculates at once all the couplings $F_{g,n}$ which in turn can be recovered by picking a particular power of ϵ and $\bar{\epsilon}$.⁷ After a careful analysis of (4.6), one shows that the generating function is independent of $\bar{\epsilon}$. Indeed, it is given by

$$F(\epsilon) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \left(\frac{2\pi\epsilon\bar{\eta}^3}{\bar{\theta}_1(\pi\bar{\epsilon})} \right)^2 e^{-\frac{\pi}{\tau_2} \bar{\epsilon}^2} \frac{1}{\eta^4 \bar{\eta}^{24}} \frac{1}{2} \sum_{h,g=0}^1 G^{\text{ferm}}[g](\check{\epsilon}_+) Z[g]^h \Gamma_{(2,2+8)}(T, U, Y). \tag{4.7}$$

The details of the notation and the derivation can be found in [1]. In particular, expanding in the deformation parameters, one finds that

$$\boxed{F_{g,n} = \delta_{n,0} F_g}, \tag{4.8}$$

with F_g denoting here the heterotic one-loop topological amplitude.

5. Conclusions

In the present paper, I have realised the $\bar{\Omega}$ -deformation, the complex conjugate of the Ω -deformation, in string theory in terms of a physical field in the string spectrum. In type I string theory compactified on $T^2 \times K3$, that is the vector partner of the S' scalar which describes

⁷ The details of the combinatorics that relate $F_{g,n}$ to the generating function F can be found in [1].

the coupling of the D5-branes. By coupling this field to the graviphoton as well as to all the massless degrees of freedom of the D5-D9 system, I have derived the deformed Yang-Mills and ADHM effective actions. As already noticed in [1], it is crucial to include a quadratic deformation which corresponds to giving a background to the S' scalar at quadratic order in the momenta. This proves that the combination of graviphoton and S' -vector, together with the quadratic background, is a correct description of the full Ω -background.

Furthermore, as already shown in [1], one can go beyond the pure field theory analysis by calculating, in the dual heterotic theory, the topological amplitudes F_g deformed by the Ω -deformation. Surprisingly, not only one recovers the perturbative part of the Nekrasov partition function, but also the full string result is independent of $\bar{\Omega}$. It would be interesting to understand this statement purely at the string level as a Q -exactness of some operator, as in the gauge theory.

From the conceptual point of view, the present analysis shows that the graviphoton differs from the Ω -deformation. Indeed, it is clear that the quadratic deformation is essential to give a correct description of the full non-holomorphic Ω -deformation. Yet, this additional piece corresponds to a different field in the string spectrum. However, Ω and the graviphoton agree at linear order and this is why, practically, one can neglect this subtlety.

As a natural application, it would be interesting to generalise my study beyond the topological limit. This is a priori tedious since, generically, the full Ω -deformation in string theory breaks topological invariance. Yet, simplifications must occur once the complete $\bar{\Omega}$ -deformation is included since the field theory limit is purely holomorphic. It is not clear, however, that the decoupling at the string level would still hold.

Acknowledgements

I would like to thank O. Schlotterer for useful discussions. This work was supported by the Swiss National Science Foundation under grant number 175641.

Appendix A. Notations and conventions

A.1. Spinors

I present some of our notations and conventions. $SO(4)$ spinor indices are raised and lowered using the standard epsilon tensors

$$\epsilon^{12} = \epsilon_{12} = 1, \quad \epsilon^{\dot{1}\dot{2}} = \epsilon_{\dot{1}\dot{2}} = -1, \quad (\text{A.1})$$

such that

$$\psi^a = +\epsilon^{ab}\psi_b, \quad \psi_a = -\epsilon_{ab}\psi^b, \quad \psi^{\dot{a}} = -\epsilon^{\dot{a}\dot{b}}\psi_{\dot{b}}, \quad \psi_{\dot{a}} = +\epsilon_{\dot{a}\dot{b}}\psi^{\dot{b}}. \quad (\text{A.2})$$

In addition, σ -matrices for $SO(4)$, $(\sigma^k)_{a\dot{a}}$ and $(\bar{\sigma}^k)^{\dot{a}a}$, are defined as

$$\sigma^\mu = (\mathbb{1}, -i\sigma), \quad \bar{\sigma}^k = (\mathbb{1}, +i\sigma), \quad (\text{A.3})$$

and are related by transposition. On the other hand, I denote S_α (resp. $S_{\dot{\alpha}}$) the self-dual (resp. anti-self-dual) spin fields of the space-time $SO(4)$. The spin fields for the internal manifold are, instead, $S_A, S^A, S_{\hat{A}}, S^{\hat{A}}$. Notice that covariant and contravariant indices (A, \hat{A}) of $SO(2)_\pm$ reflect charges $\pm 1/2$ with respect to $SO(2)$ according to the decomposition $SO(6) \rightarrow SO(2) \times SO(4)$. Our conventions for the internal spin fields can be found in the table below.

Spin field	$SO(2)$	$SO(4)$
S_A	+	$(--), (++)$
S^A	-	$(++), (--)$
$S_{\hat{A}}$	-	$(-+), (+-)$
$S^{\hat{A}}$	+	$(+-), (-+)$

(A.4)

In addition, the Lorentz generators $\sigma^{\mu\nu}, \bar{\sigma}^{\mu\nu}$ of $SO(4)_{ST}$ are

$$(\sigma_{\mu\nu})_{\alpha}{}^{\beta} \equiv \frac{1}{2}(\sigma_{\mu}\bar{\sigma}_{\nu} - \sigma_{\nu}\bar{\sigma}_{\mu})_{\alpha}{}^{\beta}, \quad (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{1}{2}(\bar{\sigma}_{\mu}\sigma_{\nu} - \bar{\sigma}_{\nu}\sigma_{\mu})^{\dot{\alpha}}{}_{\dot{\beta}}. \quad (A.5)$$

They are symmetric in the spinor indices $(\sigma_{\mu\nu})_{\alpha\beta} = +(\sigma_{\mu\nu})_{\beta\alpha}$ and $(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} = +(\bar{\sigma}_{\mu\nu})_{\dot{\beta}\dot{\alpha}}$. Also, they are (anti-)self-dual in the sense

$$(\sigma^{\mu\nu})_{\alpha\beta} = +\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\sigma_{\rho\sigma})_{\alpha\beta}, \quad (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\sigma}_{\rho\sigma})_{\dot{\alpha}\dot{\beta}}. \quad (A.6)$$

Therefore, I can use them to define (anti-)self-dual tensors. For example, for the self-dual $F_{\mu\nu}^{(+)}$ and anti-self-dual $F_{\mu\nu}^{(-)}$ part of the field strength tensor of a given gauge field, I write

$$F_{\dot{\alpha}\dot{\beta}}^{(+)} \equiv (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} F_{\mu\nu}^{(+)}, \quad F_{\alpha\beta}^{(-)} \equiv (\sigma^{\mu\nu})_{\alpha\beta} F_{\mu\nu}^{(-)}. \quad (A.7)$$

Indeed, $F_{\mu\nu}^{(\pm)} = \mp \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(F^{(\pm)})^{\rho\sigma}$. Also, note the following useful identity:

$$\sigma^{\mu\nu}\sigma^{\rho\sigma} = -(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} + \epsilon^{\mu\nu\rho\sigma})\text{Id}_2 + (\delta^{\sigma[\mu}\sigma^{\nu]\rho} - \delta^{\rho[\mu}\sigma^{\nu]\sigma}), \quad (A.8)$$

from which I can derive the following identities involving the traces of several Lorentz generators. For instance, for three generators, one has

$$\begin{aligned} & \text{Tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}\sigma^{\lambda\delta}) \\ &= 2\delta^{\sigma[\mu}(\delta^{\nu]\delta}\delta^{\rho\lambda} - \delta^{\nu]\lambda}\delta^{\rho\delta} - \epsilon^{\nu]\rho\lambda\delta}) - 2\delta^{\rho[\mu}(\delta^{\nu]\delta}\delta^{\sigma\lambda} - \delta^{\nu]\lambda}\delta^{\sigma\delta} - \epsilon^{\nu]\sigma\lambda\delta}), \end{aligned} \quad (A.9)$$

whereas for four generators I show that

$$\begin{aligned} & \text{Tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}\sigma^{\lambda\delta}\sigma^{\kappa\tau}) \\ &= 2(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} + \epsilon^{\mu\nu\rho\sigma})(\delta^{\lambda\kappa}\delta^{\delta\tau} - \delta^{\lambda\tau}\delta^{\delta\kappa} + \epsilon^{\lambda\delta\kappa\tau}) \\ &+ 2\delta^{\sigma[\mu}(\delta^{\nu]\kappa}\delta^{\rho[\delta} - \delta^{\nu]\delta}\delta^{\rho\kappa} + \epsilon^{\nu]\rho\kappa[\delta})\delta^{\lambda]\tau} \\ &+ 2\delta^{\rho[\mu}(\delta^{\nu]\tau}\delta^{\sigma[\delta} - \delta^{\nu]\delta}\delta^{\sigma\tau} + \epsilon^{\nu]\sigma\tau[\delta})\delta^{\lambda]\kappa} \\ &- 2\delta^{\rho[\mu}(\delta^{\nu]\kappa}\delta^{\sigma[\delta} - \delta^{\nu]\delta}\delta^{\sigma\kappa} + \epsilon^{\nu]\sigma\kappa[\delta})\delta^{\lambda]\tau} \\ &- 2\delta^{\sigma[\mu}(\delta^{\nu]\tau}\delta^{\rho[\delta} - \delta^{\nu]\delta}\delta^{\rho\tau} + \epsilon^{\nu]\rho\tau[\delta})\delta^{\lambda]\kappa}. \end{aligned} \quad (A.10)$$

Finally, it is useful to define the self-dual and anti-self-dual 't Hooft symbols by decomposing the sigma-matrices, with $c = 1, 2, 3$:

$$(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv i\bar{\eta}_{\mu\nu}^c(\sigma_c)^{\dot{\alpha}}{}_{\dot{\beta}}, \quad (\sigma_{\mu\nu})_{\alpha}{}^{\beta} \equiv i\eta_{\mu\nu}^c(\sigma_c)_{\alpha}{}^{\beta}. \quad (A.11)$$

A.2. Operator product expansions

The operator product expansion algebra for the ten-dimensional fields can be decomposed according to the compactified theory. Indeed, the space-time current algebra is

$$S^{\dot{\alpha}}(z)S_{\beta}(w) \sim \frac{1}{\sqrt{2}}(\bar{\sigma}^{\mu})^{\dot{\alpha}}_{\beta}\psi_{\mu}(w), \quad S_{\alpha}(z)S^{\dot{\beta}}(w) \sim \frac{1}{\sqrt{2}}(\sigma^{\mu})_{\alpha}^{\dot{\beta}}\psi_{\mu}(w), \quad (\text{A.12})$$

$$S^{\dot{\alpha}}(z)S^{\dot{\beta}}(w) \sim -\frac{\epsilon^{\dot{\alpha}\dot{\beta}}}{(z-w)^{1/2}}, \quad S_{\alpha}(z)S_{\beta}(w) \sim \frac{\epsilon_{\alpha\beta}}{(z-w)^{1/2}}, \quad (\text{A.13})$$

$$\psi^{\mu}(z)S^{\dot{\alpha}}(w) \sim \frac{1}{\sqrt{2}}\frac{(\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}S_{\beta}(w)}{(z-w)^{1/2}}, \quad \psi^{\mu}(z)S_{\alpha}(w) \sim \frac{1}{\sqrt{2}}\frac{(\sigma^{\mu})_{\alpha\dot{\beta}}S^{\dot{\beta}}(w)}{(z-w)^{1/2}}, \quad (\text{A.14})$$

$$J^{\mu\nu}(z)S^{\dot{\alpha}}(w) \sim -\frac{1}{2}\frac{(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\beta}S^{\dot{\beta}}(w)}{z-w}, \quad J^{\mu\nu}(z)S_{\alpha}(w) \sim -\frac{1}{2}\frac{(\sigma^{\mu\nu})_{\alpha}^{\beta}S_{\beta}(w)}{z-w}. \quad (\text{A.15})$$

As for the internal one, it is given by

$$S^A(z)S_B(w) \sim \frac{i\delta^A_B}{(z-w)^{3/4}}, \quad S_A(z)S^B(w) \sim \frac{i\delta_A^B}{(z-w)^{3/4}} \quad (\text{A.16})$$

$$S^A(z)S^B(w) \sim \frac{i}{\sqrt{2}}\frac{(\Sigma^I)^{AB}\psi_I}{(z-w)^{1/4}}, \quad S_A(z)S_B(w) \sim -\frac{i}{\sqrt{2}}\frac{(\Sigma^I)_{AB}\psi_I}{(z-w)^{1/4}}, \quad (\text{A.17})$$

$$\psi^I(z)S_A(w) \sim \frac{1}{\sqrt{2}}\frac{(\bar{\Sigma}^I)_{AB}S^B(w)}{(z-w)^{1/2}}, \quad \psi^I(z)S^A(w) \sim -\frac{1}{\sqrt{2}}\frac{(\Sigma^I)^{AB}S_B(w)}{(z-w)^{1/2}}, \quad (\text{A.18})$$

$$J^{IJ}(z)S^A(w) \sim \frac{1}{2}\frac{(\bar{\Sigma}^{IJ})^A_B S^B(w)}{z-w}, \quad J^{IJ}(z)S_A(w) \sim \frac{1}{2}\frac{(\Sigma^{IJ})_A^B S_B(w)}{z-w}. \quad (\text{A.19})$$

Using the algebras above, one easily derives all the necessary correlation functions used throughout the manuscript. For instance, the correlation function an internal fermion with two spin fields is

$$\langle \Psi(z_1)S_{\hat{A}}(z_2)S_{\hat{B}}(z_3) \rangle = \frac{\epsilon_{\hat{A}\hat{B}}}{z_{12}^{\frac{1}{2}}z_{13}^{\frac{1}{2}}z_{23}^{\frac{1}{4}}}, \quad (\text{A.20})$$

whereas the correlator of two space-time fermions with two spin-fields is

$$\langle \psi^{\mu}(z_1)\psi^{\nu}(z_2)S_{\alpha}(z_3)S_{\beta}(z_4) \rangle = -\frac{1}{2}\frac{\sigma^{\mu\nu}_{\alpha\beta}z_{34}^{\frac{1}{2}}}{(z_{13}z_{14}z_{23}z_{24})^{\frac{1}{2}}} + \frac{1}{2}\frac{\delta^{\mu\nu}\epsilon_{\alpha\beta}(z_{13}z_{24} + z_{14}z_{23})}{z_{12}(z_{13}z_{14}z_{23}z_{24}z_{34})^{\frac{1}{2}}}. \quad (\text{A.21})$$

Another useful result is the correlation function of four spin fields:

$$\langle S_{\alpha}(z_1)S_{\beta}(z_2)S_{\gamma}(z_3)S_{\delta}(z_4) \rangle = \frac{\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}z_{14}z_{23} - \epsilon_{\alpha\delta}\epsilon_{\beta\gamma}z_{13}z_{24}}{(z_{12}z_{13}z_{14}z_{23}z_{24}z_{34})^{\frac{1}{2}}}. \quad (\text{A.22})$$

If one additionally inserts a space-time current, then

$$\begin{aligned} & \langle \psi^{\mu}\psi^{\nu}(z_1)S_{\alpha}(z_2)S_{\beta}(z_3)S_{\gamma}(z_4)S_{\delta}(z_5) \rangle \\ &= \frac{1}{2}\frac{\epsilon_{\beta\gamma}(\sigma^{\mu\nu})_{\alpha\delta}z_{14}z_{15}z_{36} + \epsilon_{\alpha\delta}(\sigma^{\mu\nu})_{\beta\gamma}z_{13}z_{16}z_{45}}{z_{12}z_{13}z_{14}z_{15}(z_{23}z_{24}z_{25}z_{34}z_{35}z_{45})^{\frac{1}{2}}}. \end{aligned} \quad (\text{A.23})$$

Correlation functions for higher numbers of fermionic fields can be found in [17].

Finally, the two-point function for bosonic fields with Dirichlet boundary conditions is

$$\langle \partial Z_{DD}(z_1) \partial \bar{Z}_{DD}(z_2) \rangle = \frac{1}{z_{12}^2}, \quad (\text{A.24})$$

$$\langle \partial Z_{DD}(z_1) \bar{\partial} \bar{Z}_{DD}(z_2) \rangle = -\frac{1}{z_{12}^2}, \quad (\text{A.25})$$

$$\langle \bar{\partial} Z_{DD}(z_1) \bar{\partial} \bar{Z}_{DD}(z_2) \rangle = \frac{1}{z_{12}^2}, \quad (\text{A.26})$$

which can be found using

$$\langle Z_{DD}(z_1) \bar{Z}_{DD}(z_2) \rangle = \log |z_{12}|^2 - \log |\bar{z}_{12}|^2. \quad (\text{A.27})$$

For the N-N directions, the same result holds with positive sign for all the two-point functions. For the fermions in the NS sector, it is the opposite:

$$\langle \psi_{NN}(z_1) \psi_{NN}(z_2) \rangle = \frac{1}{z_{12}}, \quad \langle \psi_{NN}(z_1) \psi_{NN}(\bar{z}_2) \rangle = -\frac{1}{z_{12}}, \quad (\text{A.28})$$

$$\langle \psi_{DD}(z_1) \psi_{DD}(z_2) \rangle = \frac{1}{z_{12}}, \quad \langle \psi_{DD}(z_1) \psi_{DD}(\bar{z}_2) \rangle = \frac{1}{z_{12}}. \quad (\text{A.29})$$

References

- [1] K.S. Narain, A. Zein Assi, The decoupling of $\bar{\Omega}$ in string theory, arXiv:1807.11432 [hep-th].
- [2] A. Losev, N. Nekrasov, S.L. Shatashvili, Testing Seiberg-Witten solution, in: Strings, Branes and Dualities. Proceedings, NATO Advanced Study Institute, Cargese, France, May 26–June 14, 1997, pp. 359–372.
- [3] N.A. Nekrasov, Seiberg-Witten prepotential from instanton counting, Adv. Theor. Math. Phys. 7 (5) (2003) 831–864, arXiv:hep-th/0206161.
- [4] N. Nekrasov, A. Okounkov, Seiberg-Witten theory and random partitions, Prog. Math. 244 (2006) 525–596, arXiv:hep-th/0306238.
- [5] I. Antoniadis, E. Gava, K.S. Narain, T.R. Taylor, $N = 2$ type II – Heterotic duality and higher derivative F -terms, Nucl. Phys. B 455 (1–2) (1995) 109–130, arXiv:hep-th/9507115 [hep-th].
- [6] I. Antoniadis, E. Gava, K.S. Narain, T.R. Taylor, Topological amplitudes in string theory, Nucl. Phys. B 413 (1994) 162–184, arXiv:hep-th/9307158.
- [7] I. Antoniadis, I. Florakis, S. Hohenegger, K.S. Narain, A. Zein Assi, Worldsheet realization of the refined topological string, Nucl. Phys. B 875 (2013) 101–133, arXiv:1302.6993 [hep-th].
- [8] I. Antoniadis, I. Florakis, S. Hohenegger, K.S. Narain, A. Zein Assi, Non-perturbative Nekrasov partition function from string theory, Nucl. Phys. B 880 (2014) 87–108, arXiv:1309.6688 [hep-th].
- [9] I. Antoniadis, I. Florakis, S. Hohenegger, K.S. Narain, A. Zein Assi, Probing the moduli dependence of refined topological amplitudes, Nucl. Phys. B 901 (2015) 252–281, arXiv:1508.01477 [hep-th].
- [10] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Commun. Math. Phys. 165 (1994) 311–428, arXiv:hep-th/9309140.
- [11] M. Billo, M. Frau, F. Fucito, A. Lerda, Instanton calculus in R-R background and the topological string, J. High Energy Phys. 11 (2006) 012, arXiv:hep-th/0606013 [hep-th].
- [12] K. Ito, H. Nakajima, T. Saka, S. Sasaki, $N = 2$ instanton effective action in Ω -background and D3/D(–1)-brane system in R-R background, J. High Energy Phys. 11 (2010) 093, arXiv:1009.1212 [hep-th].
- [13] I. Antoniadis, H. Partouche, T.R. Taylor, Duality of $N = 2$ heterotic type I compactifications in four-dimensions, Nucl. Phys. B 499 (1997) 29–44, arXiv:hep-th/9703076 [hep-th].
- [14] I. Antoniadis, H. Partouche, T.R. Taylor, Lectures on heterotic-type I duality, Nucl. Phys. Proc. Suppl. 61A (1998) 58–71, arXiv:hep-th/9706211 [hep-th], 1997, [179 (1997)].
- [15] M.B. Green, M. Gutperle, D-instanton induced interactions on a D3-brane, J. High Energy Phys. 02 (2000) 014, arXiv:hep-th/0002011 [hep-th].

- [16] A. Zein Assi, Topological Amplitudes and the String Effective Action, PhD thesis, CERN, 2013, arXiv:1402.2428 [hep-th].
- [17] D. Haertl, O. Schlotterer, S. Stieberger, Higher point spin field correlators in $D = 4$ superstring theory, Nucl. Phys. B 834 (2010) 163–221, arXiv:0911.5168 [hep-th].
- [18] S. Stieberger, Open & closed vs. pure open string disk amplitudes, arXiv:0907.2211 [hep-th].
- [19] M. Moskovic, A. Zein Assi, Instanton corrections for m and Ω , Nucl. Phys. B 920 (2017) 601–619, arXiv:1612.06414 [hep-th].